Symplectic geometry of scattering diagrams for log CY surfaces

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Outline

1. Log CY surfaces
2. Floer-theoretic reconstruction
3. Asymptotic disk counts
4. Consequences

Log CY surfaces

Definition

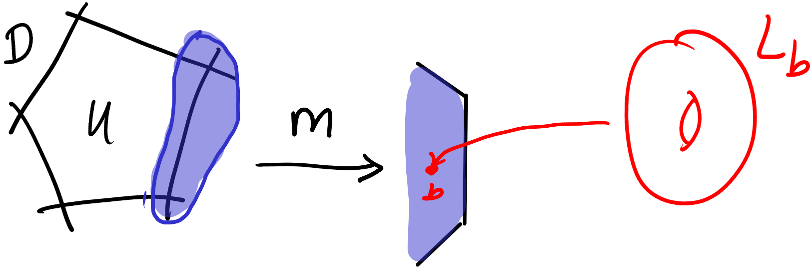
A (exact) log CY surface (with maximal boundary) is a pair (Y, D)

- Y smooth projective surface,
- D nodal anticanonical divisor, which supports an ample divisor.

Thus $U = Y \setminus D$ is an exact symplectic manifold.
Example: $Y =$ cubic surface, $D =$ triangle of lines.

Topology near D

The divisor D consists of a cycle of $\mathbb{P}^1$'s. We can arrange that the symplectic structure is *locally toric* near D. Fibers of local moment maps give Lagrangian tori L_b near D. L_b has Maslov class zero in U.



Reconstruction

[Fukaya, Tu, Abouzaid-Auroux-Katzarkov] For generic J, L_b bounds no disks (since space of disks with boundary on L_b has virtual dimension $\mu + n - 3 = -1$), and its Floer theory is undeformed. The space of objects supported on L_b is therefore identified with $H^1(L_b, U_\Lambda) \cong U_\Lambda^2$.

$$\Lambda = \left\{ \sum_{i=1}^{\infty} c_i T^{r_i} \mid c_i \in \mathbb{C}, r_i \in \mathbb{R}, \lim_{i \to \infty} r_i = \infty \right\}$$

$$U_\Lambda = \left\{ c_0 + \sum_{i=1}^{\infty} c_i T^{r_i} \mid c_0 \neq 0, r_i > 0 \right\} \subset \Lambda$$

Reconstruction

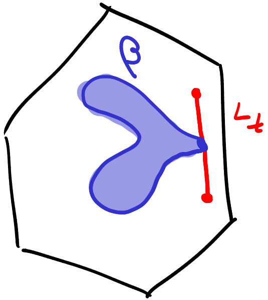
As L_b moves, glue together $H^1(L_b, U_\Lambda)$ by a combination of *rescalings* and *maps induced by pseudoisotopies of A_∞-structures*.

- Rescalings: (L_t, J_t), L_t non-hamiltonian deformation, but J_t chosen so that no disks appear. This just rescales the coefficients.
- Pseudoisotopies: (L_t, J_t), L_t hamiltonian deformation, but as J_t varies Maslov 0 disks may appear; in this parametrized problem they have dimension $\mu + n - 3 + 1 = 0$.

The pseudoisotopies realize a wall-crossing phenomenon (walls of the first kind).

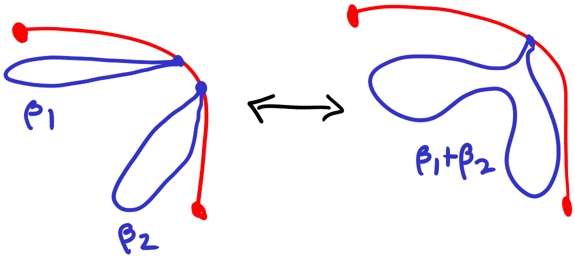
Reconstruction

- Let $(z_1, z_2) \in U_\Lambda^2$ be coordinates from an identification $H^1(L_0, U_\lambda) \cong U_\Lambda^2$.
- In the case where all disks in the family (L_t, J_t) lie in multiples of a particular class $\beta \in H_2(X, L_0)$, the pseudoisotopy acts as a transformation $z_i \mapsto h_i(z_\beta) z_i$, where $h_i(z) \in 1 + z\mathbb{Q}[[z]]$ and $z_\beta = T^{\omega(\beta)} z^{[\partial\beta]}$.



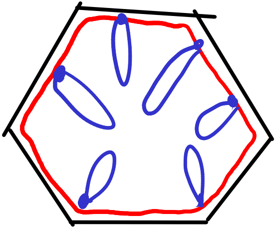
Basic problem

So, we "just" need to count the Maslov 0 disks. But these counts themselves are not stable: if we deform the path (L_t, J_t), the Maslov 0 disk counts may change (walls of the second kind). A J_t-disk can break into a $J_{t'}$-disk and a $J_{t''}$-disk. (But sphere bubbling cannot occur due to exactness of U.) [Cf. Yu-Shen Lin]



Tori near boundary

Take path $b(t)$ in base of torus fibration near D, and consider a path of Lagrangian tori $L_{b(t)}$ close to the boundary divisor D. We want to apply Fukaya's procedure to this family of tori. After a small reformulation, we need to count $\mu = 0$ disks with boundary on $L_{b(t)}$ for some t.



Stabilization

Lemma

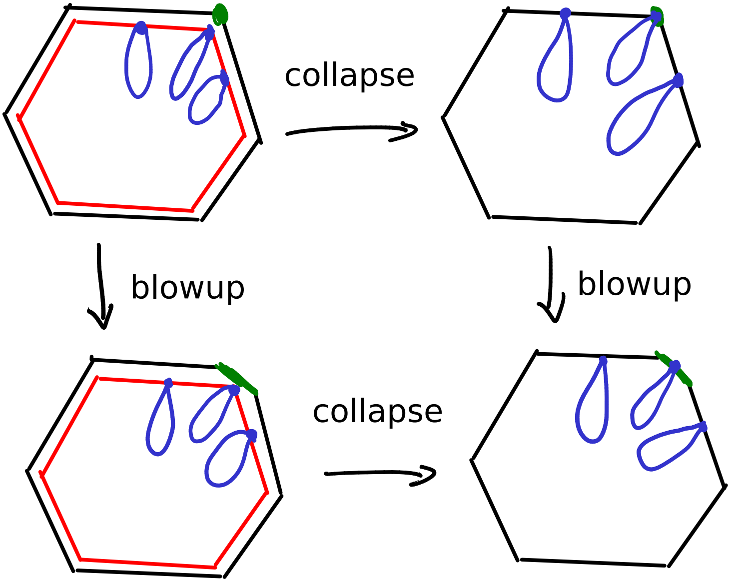
The moduli spaces of $\mu = 0$ disks stablize as the path $b(t)$ approaches the boundary of the moment map image.

- That is to say, for each relative homotopy class $\beta \in H_2(X, L_{b(t)})$, if we take the path $b(t)$ close enough to the boundary, eventually no wall-crossing of the second kind occurs.
- Morally, this is related to the idea that the GHK scattering diagram is an asymptotic object.

Stabilization

- As the path of tori $L_{b(t)}$ collapses onto the boundary divisor, holomorphic disks must approach holomorphic spheres in Y (target-local Gromov compactness [J.W. Fish]).
- By successively blowing up nodes of D, one can assume that the disks in the class β we are interested in are limiting to spheres that intersect the smooth part of D. Eventually, no bifurcations can occur because disks in classes β_1, β_2 such that $\beta = \beta_1 + \beta_2$ are separated from each other.

Stabilization



Relative invariants

- Let $(\tilde{Y}, \tilde{D})$ be a toric blow up of (Y, D), C a component of $\tilde{D}$, $\tilde{D}' = \overline{\tilde{D} \setminus C}$. Set $\tilde{Y}^0 = Y \setminus \tilde{D}'$, $C^0 = C \cap \tilde{Y}^0$.
- Let $\beta \in H_2(\tilde{Y}, \mathbb{Z})$ be a class such that $\beta \cdot C = k_\beta$ and $\beta \cdot (\text{each other component}) = 0$.
- Consider space of relative maps $\overline{\mathcal{M}}(\tilde{Y}/C, \beta)$ with full tangency of order k_β and an unspecified point of C^0, and the subspace $\overline{\mathcal{M}}(\tilde{Y}^0/C^0, \beta)$ of maps whose image lies in $\tilde{Y}^0$.

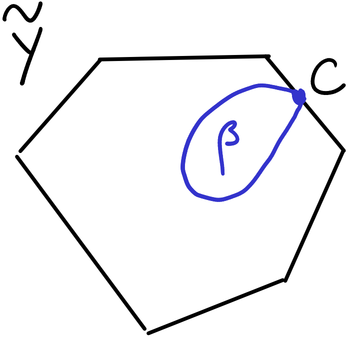
Lemma (Gross-Pandharipande-Siebert, GHK)

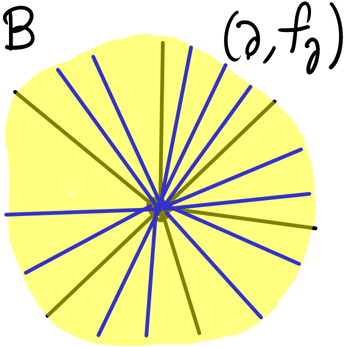
$\overline{\mathcal{M}}(\tilde{Y}^0/C^0, \beta)$ is compact.

- Define $N_\beta = \#\overline{\mathcal{M}}(\tilde{Y}^0/C^0, \beta)$ to be the virtual count of curves.
- The GHK scattering diagram is defined on an singular affine manifold B in terms of N_β. The rays correspond to components in some toric blow up.

Scattering diagram

$$f(z) = \exp\left[\sum_{\beta} k_\beta N_\beta T^{\omega(\beta)} z^{[\partial\beta]}\right], z_i \mapsto f(z)^{n_i} z_i$$



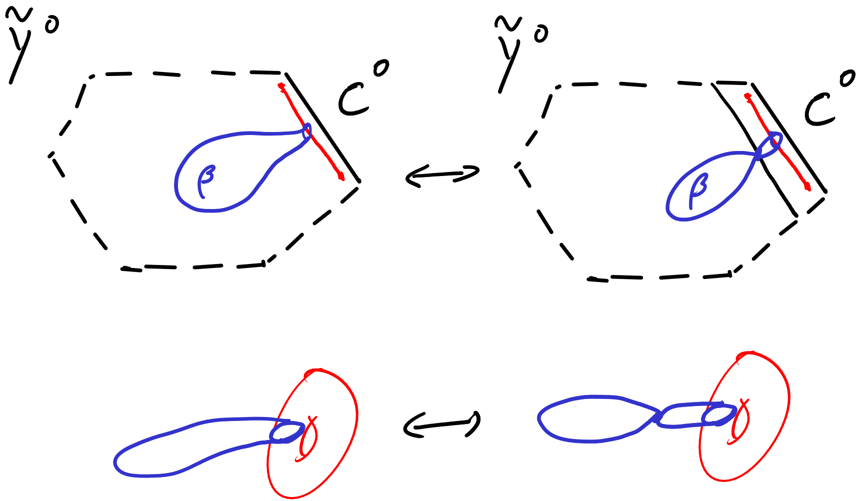


Symplectic sum

- To understand disks in $\tilde{Y}^0$, perform a symplectic cut on a neighborhood of C^0, thus writing $\tilde{Y}^0 = \tilde{Y}^0 \#_{C^0} \mathbb{P}(N_{C^0} \oplus \mathbb{C})$.
- Perform the cut in such a way that a piece of the Lagrangian boundary condition L_t remains in the $\mathbb{P}(N_{C^0} \oplus \mathbb{C})$ factor.
- A symplectic sum formula relates the parameterized moduli spaces of disks with boundary on L_t to (relative) moduli spaces of disks in $\mathbb{P}(N_{C^0} \oplus \mathbb{C})$ and moduli spaces of closed curves in $\tilde{Y}^0$.
- The closed curves are precisely the ones used in the definition of the GHK scattering diagram!

[Li-Ruan, Ionel-Parker, Jun Li]
[Nishinou-Siebert, M. Farajzedeh Tehrani]

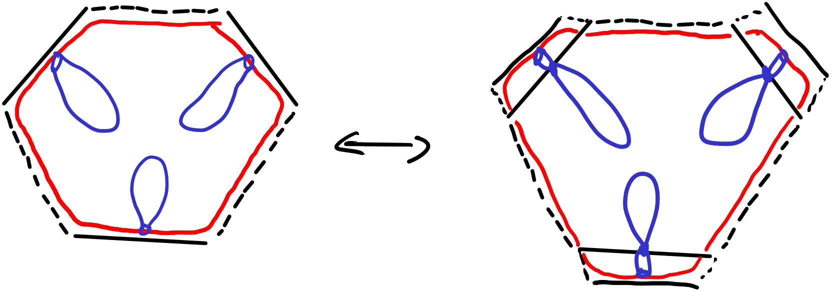
Symplectic sum



Compatibility

- Since the disk counts are not entirely stable, it is important to understand whether different degenerations are compatible.
- To deal with several divisor components ($\leftrightarrow$ rays) simultaneously, perform toric blow-ups to make the divisors we are interested in disjoint.
- We can treat this situation as symplectic sum along a disconnected divisor.

Compatibility



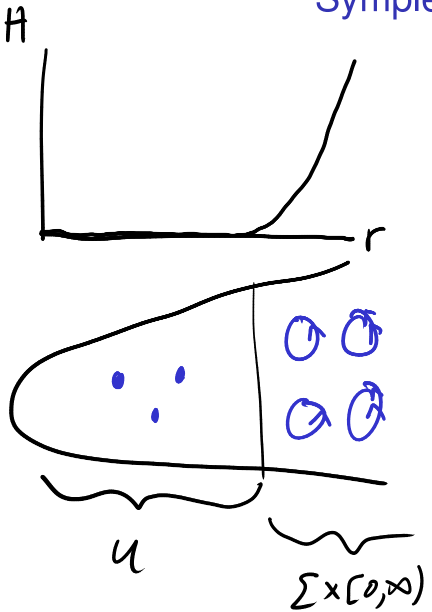
Compatibility

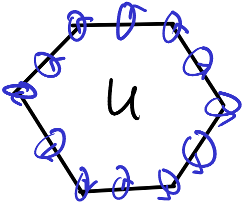
- In this way, we can show that the disk counts match the GHK formula for any particular finite collection of rays in the scattering diagram.
- Work order by order in Novikov parameter T: To a particular order, all but finitely many rays are trivial.
- Thus, by working with more and more “refined” degenerations, we can calculate the wall-crossing to any desired order, and match it to the GHK description.
- Thus, for families of Lagrangian tori $L_{b(t)}$ sufficiently close to the boudary divisor, the wall crossing transformations are given by the functions in the GHK scattering diagram.

Symplectic cohomology

- A Floer cohomology for Liouville manifolds, such as $U = Y \setminus D$.
- Complete U to $\widehat{U}$ by attaching a positive half of a symplectization $\Sigma \times [0, \infty)$, $\Sigma \subset U$ a contact type hypersurface.
- Let H be a Hamiltonian that grows like cr^2 on the end, define $SH^*(U) = HF(H)$. (Actually involves perturbation of H.)
- $SH^*(U)$ detects classical topology of U and the closed Reeb orbits in Σ, in a deformation-invariant way.

Symplectic cohomology





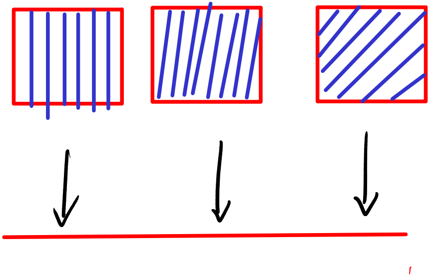
Basis of SH

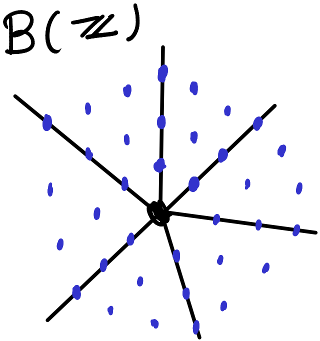
- In the log CY surface case, Σ is a T^2-bundle over S^1 with nontrivial monodromy.
- The Reeb flow is tangent to the T^2 fibers, and translates each one by a certain vector $v(t) \in \mathbb{R}^2$. One can arrange that the vector $v(t)$ rotates monotonically as we vary the fiber.
- When $v(t)$ has rational slope, periodic orbits appear. These orbits and their iterates contribute non-trivial cocycles to symplectic cohomology.

Theorem (P. 2013)

There is a basis for the degree zero symplectic cohomology $SH^0(U)$ indexed by the integral points of the GHK affine manifold B.

Basis of SH



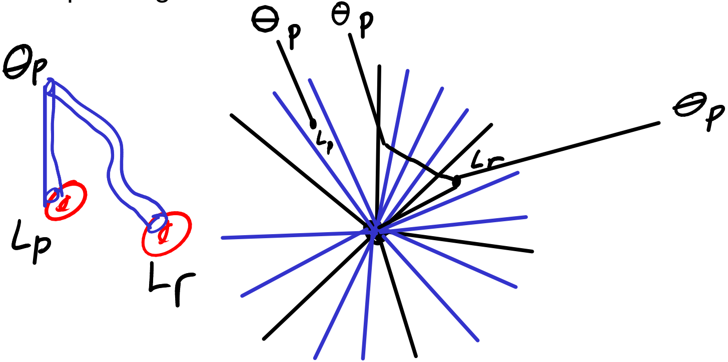


Closed-open map

- The representation of elements of $SH^0(U)$ as functions on the mirror is furnished by the closed-open map counting half-cylinders asymptotic to SH generators with boundary on Lagrangian tori, weighted by area and holonomy. $SH^0(U) \to \mathcal{O}(H^1(L, U_\Lambda))$.
- For any $p \in B(\mathbb{Z})$, we have a divisor component C_p (in some toric blow up $(\tilde{Y}, \tilde{D})$), we have a torus L_p and a generator $\theta_p \in SH^0(U)$ linking that divisor.
- In some situations, we can argue that the closed open map sends θ_p to a monomial in $\mathcal{O}(H^1(L_p, U_\lambda))$. (One which is invariant w.r.t. wall-crossing along C).

Closed-open map

- From here on, we can follow GHK formally.
- Image of θ_p in other charts $\leftrightarrow$ broken lines.
- To determine the coefficient of θ_r in $\theta_p \cdot \theta_q$, evaluate in the chart corresponding to L_r, and take the coefficient of the corresponding monomial.



Conclusion

- Construction of the mirror is governed by $\mu = 0$ disks, the counts of which stabilize in the limit as the tori collapse onto the divisor.
- These asymptotic disk counts are related to relative GW invariants for closed $g = 0$ curves via a symplectic sum formula.
- This gives us enough information to construct the mirror "at infinity," and to understand other structures, such as $SH^0(U)$, that also live "at infinity."

End

Thank you!